Section 15.6

Tangent Planes

We wish to find a plane tangent to a surface $z = f(x, y)$ at a point $(a, b, f(a, b))$ for differentiable f.

As a tangent plane should be tangent to the surface in all directions, begin by computing the tangent vector of every curve in the surface $z = f(x, y)$ as it passes through $(a, b, f(a, b))$.

Tangent Plane for $z = f(x, y)$
Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

Example 15.6.1. Find an equation of the plane tangent to the paraboloid $z = f(x, y) = 32 - 3x^2 - 4y^2$ at $(2, 1, 16)$.

Linear Approximation

How close is our tangent plane equation to the differentiability criterion?

Example 15.6.2. Approximate $f(-1.05, 2.1)$ when $f(x, y) = \dfrac{5}{x^2 + y^2}$.

For a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with linear approximation $L(x) \approx f(x)$ near $x = a$, the change in $y = L(x)$ is the differential of y denoted by dy (the change in x is here denoted by dx).

As L is linear, it follows that

$$dy = f'(a)dx.$$

Definition: Total Differential
For a differentiable function $z = f(x, y)$, the change in the linear approximation $L(x, y) \approx f(x, y)$ near $(x, y) = (a, b)$ is given by

$$dz = f_x(a, b)dx + f_y(a, b)dy.$$

We call dz the **total differential**.

Example 15.6.3. Approximate the change of $f(x, y) = \dfrac{5}{x^2 + y^2}$ when the independent variable changes from $(-1, 2)$ to $(-0.93, 1.94)$.

NB while the formula for dz includes both the differentials dx and dy, dz is nevertheless a differential 1-form. It can then only be used to measure z change for straight line change (or approx) in (x, y). For more on differential forms, see Terence Tao's notes: https://www.math.ucla.edu/ tao/preprints/forms.pdf

More Tangent Planes

To find the plane containing all vectors tangent to a point (a, b, c) on a surface defined by $S : F(x, y, z) = 0$, we consider a curve $C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ lying on S.

Example 15.6.4. An ellipsoid is not the graph of a function of two independent variables. Find an equation of the plane tangent to the ellipsoid $S : \frac{x^2}{9}+\frac{y^2}{25}+z^2-1 = 0$ at the point $(0, 4, \frac{3}{5})$